write a program (or proceed by hand calculation) to generate the array $g$. Running the commercial program with $f$ and $g$ as separate load cases would then give the strains to be used as input to an additional small, user written program for calculating the force $F^f$ (and $F^g$) of Eq. (5).


### Reference

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# Radiative Transfer in Anisotropically Scattering Nonplanar Media


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## Nomenclature

| | |
|---|---|
| $B$ | = emissive power, $\pi I_b$ |
| $I$ | = radiation intensity |
| $I_b$ | = blackbody intensity |
| $\ell$ | = direction cosine |
| $n$ | = radius ratio, $r_2/r_1$ |
| $P$ | = phase function |
| $Q$ | = a constant |
| $r$ | = dimensionless radial direction $(\alpha+\sigma)r'$ |
| $r'$ | = radial direction |
| $X$ | = position vector |
| $\alpha$ | = absorption coefficient |
| $\beta$ | = polar angle |
| $\epsilon$ | = emissivity |
| $\lambda$ | = scattering albedo, $\sigma/(\alpha+\sigma)$ |
| $\xi$ | = $\cos\beta$ |
| $\sigma$ | = scattering coefficient |
| $\tau$ | = optical thickness, $r_2-r_1$ |
| $\phi$ | = azimuthal angle |
| $\omega$ | = solid angle |
| $\Omega$ | = unit direction vector |

*Subscripts*

| | |
|---|---|
| 1,2 | = inner and outer boundary, respectively |

## Introduction

THE exact treatment of radiative transfer in participating, particularly anisotropically scattering media requires immense effort and computation. In the past, several approximate solutions of the equation of transfer have been developed to overcome the mathematical complexity of the problem. The spherical harmonics method ($P_N$ approximation) is capable of estimating higher order approximate solutions. Bayazitoğlu and Higenyi[1,2] employed $P_3$ and $P_5$ approximations for nonplanar geometries in problems involving absorbing, emitting, and isotropically scattering media. Their results compared favorably with the existing exact solutions.

The purpose of this Note is to demonstrate, within the $P_3$ framework, the effects of anisotropic scattering in one-dimensional cylindrical and spherical geometries. To this effect, a linear phase function is used. Modest and Azad[3] showed that a slightly modified form of the linear model, used together with the differential $P_1$ approximation, yields quite accurate results in planar geometry.

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## Analysis

Consider the equation of transfer

$$\Omega\cdot\nabla I=\alpha I_b-(\alpha+\sigma)I+\frac{\sigma}{4\pi}\int_{4\pi}I(\Omega')\cdot p(\Omega,\Omega')\mathrm{d}\omega' \tag{1}$$

The phase function $p$ in Eq. (2) can be expanded in a series of Legendre polynomials[4]:

$$p(\Omega,\Omega')=\sum_{n=0}^{\infty}a_nP_n(\Omega,\Omega'),\ a_0\equiv 1 \tag{2}$$

In the following analysis, only the first two terms in the expansion are retained on grounds of simplicity. However, the spherical harmonics method can accommodate any number of terms.

In the spherical harmonics method, the angular distribution of the radiation intensity is expressed in a series of associated Legendre polynomials. The series is truncated after $N$ terms ($P_N$ approximation), and the coefficients are expressed in terms of the moments of intensity. In this respect, the extension of the method to anisotropic scattering problems is straightforward. When the phase function is expressed in the form of Eq. (2), the scattering integral in the equation of transfer is transformed into a summation term in the moments of intensity, thus enabling the complex anisotropic scattering effects to be represented and handled in a simple and efficient manner.

In the final step, the equation of transfer is approximated by a series of "moment" differential equations which are derived by multiplying it by the powers of direction cosines and integrating over a solid angle of $4\pi$.

To demonstrate the effects of anisotropic scattering in one-dimensional cylindrical and spherical geometrics, a gray medium at radiative equilibrium is considered.

### Cylindrical Geometry $(r,\theta,z)$

For cylindrical symmetry, the equation of transfer is

$$\ell_r\frac{\partial I}{\partial r}-\ell_\theta\frac{\partial I}{\partial\phi}=(1-\lambda)I_b-I+\frac{\lambda}{4\pi}\int_{4\pi}(I+a_1\ell_r)I\mathrm{d}\omega \tag{3}$$

where $\ell_r=\sin\beta\cos\phi$ and $\ell_\theta=\sin\beta\sin\phi$. After some algebraic manipulation, the moment differential equations take the form:

$$\frac{\mathrm{d}I_0}{\mathrm{d}r}=\frac{10}{3r}I_0-(10-\lambda a)I_r-\frac{10}{r}I_{rr}+\frac{35}{3}I_{rrr},\quad a\equiv a_1 \tag{4a}$$

$$\frac{\mathrm{d}I_r}{\mathrm{d}r}=(1-\lambda)(4B-I_0)-\frac{1}{r}I_r \tag{4b}$$

$$\frac{\mathrm{d}I_{rr}}{\mathrm{d}r}=\frac{2}{3r}I_0-\left(1-\frac{\lambda a}{3}\right)I_r-\frac{2}{r}I_{rr} \tag{4c}$$

$$\frac{\mathrm{d}I_{rrr}}{\mathrm{d}r}=\frac{(1-\lambda)}{3}4B+\frac{\lambda}{3}I_0+\frac{8}{5r}I_r-I_{rr}-\frac{3}{r}I_{rrr} \tag{4d}$$

where $I_0$ is the zeroth moment of intensity, and $I_r$, $I_{rr}$, and $I_{rrr}$ are the first, second, and third moments of intensity in the radial direction. Radiative equilibrium demands $I_r=Q/r$ and therefore from Eq. (4b), $I_0=4B$.

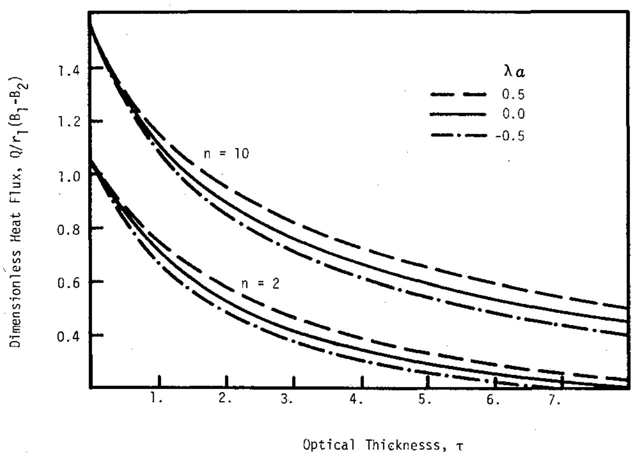


**Fig. 1 Dimensionless radiant heat flux on the inner wall vs optical thickness (concentric cylinders: $\epsilon_1 = \epsilon_2 = 1$).**

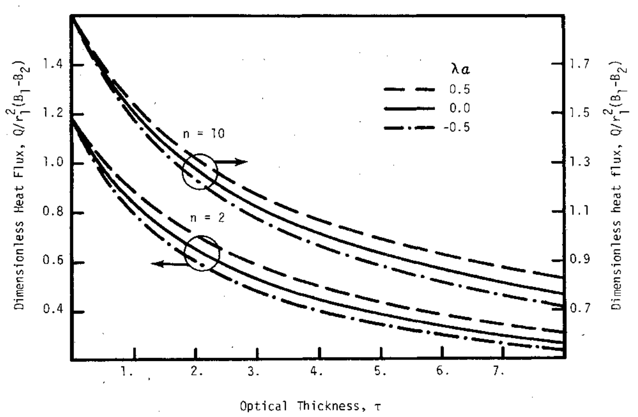


**Fig. 2 Dimensionless radiant heat flux on the inner wall vs optical thickness (concentric spheres: $\epsilon_1 = \epsilon_2 = 1$).**

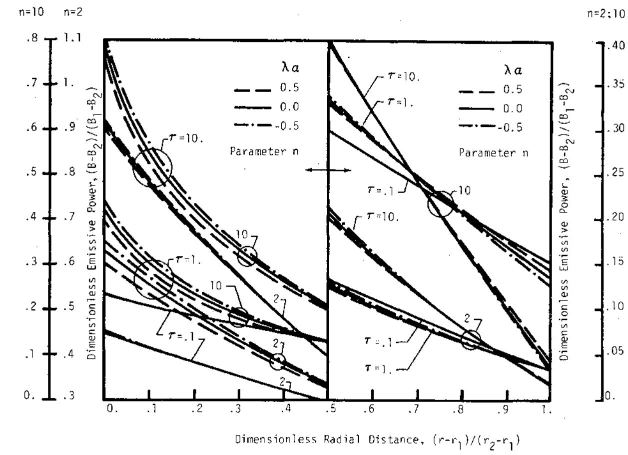


**Fig. 3 Distribution of gas emissive power between concentric cylinders.**

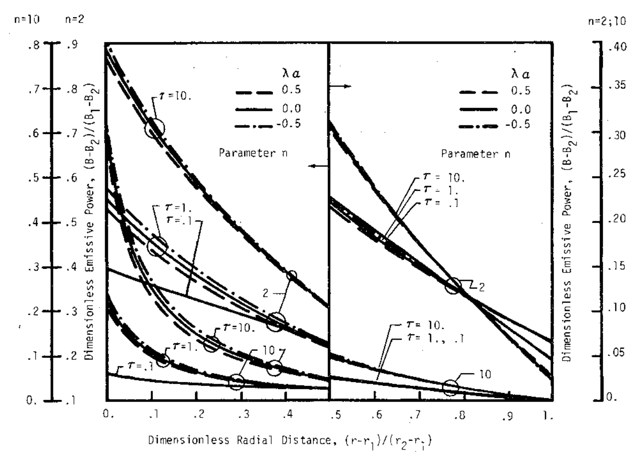


**Fig. 4 Distribution of gas emissive power between concentric spheres.**

**Spherical Geometry ($\theta$, $\varphi$, $r$)**

Under spherical geometry, the equation of transfer is

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = (1-\lambda)I_b - I + \frac{\lambda}{4\pi}\int_{4\pi}(I+a\mu)I\,d\omega \quad (5)$$

where $\mu = \cos\beta$.

The moment differential equations are

$$\frac{dI_0}{dr} = \frac{5}{r}I_0 - (10-\lambda a)I_r - \frac{15}{r}I_{rr} + \frac{35}{3}I_{rrr} \quad (6a)$$

$$\frac{dI_r}{dr} = (1-\lambda)(4B-I_0) - \frac{2}{r}I_r \quad (6b)$$

$$\frac{dI_{rr}}{dr} = \frac{1}{r}I_0 - \left(1-\frac{\lambda a}{3}\right)I_r - \frac{3}{r}I_{rr} \quad (6c)$$

$$\frac{dI_{rrr}}{dr} = \frac{(1-\lambda)}{3}4B + \frac{\lambda}{3}I_0 + \frac{2}{r}I_{rr} - I_{rr} - \frac{4}{r}I_{rrr} \quad (6d)$$

Similarly, for the case of radiative equilibrium, $I_r = Q/r^2$ and $I_0 = 4B$.

**Boundary Conditions**

The boundary conditions, derived using the Marshak approach,[5] have the same form for concentric cylindrical and spherical enclosures, and are given as[1]

$$3\epsilon_i I_0 \pm 16(2-\epsilon_i)I_r + 15\epsilon_i I_{rr} = 32\epsilon_i B_i \quad (7a)$$

$$(3\epsilon_i - 5)I_0 \pm 16(1-\epsilon_i)I_r - 15(1+\epsilon_i)I_{rr} - 32I_{rrr} = 32\epsilon_i B_i \quad (7b)$$

for diffusely reflecting boundaries, with the + and − signs applying to the inner ($i=1$) and outer ($i=2$) boundaries, respectively.

## Results and Discussion

The linear two-point boundary-value problems depicted by Eqs. (4) and (7) and by Eqs. (6) and (7) are solved by the method of particular solutions, using 500 uniformly spaced integration steps.

The dimensionless radiative flux on the inner wall of concentric cylinders and spheres with radius ratios of 2 and 10 are plotted against the optical thickness in Figs. 1 and 2, respectively. Results presented are for the isotropic ($a=0$), forward ($a>0$), and backward ($a<0$) scattering cases. Anisotropic scattering causes a maximum change of ±15% about the isotropic scattering case for concentric cylinders and a change of ±16% for concentric spheres. The shortcomings of the $P_3$ approximation (mainly small optical thicknesses and/or high radius ratios) should also be valid for the forward and backward scattering cases.

Figures 3 and 4 show the respective distributions of dimensionless emissive power in concentric cylindrical and spherical layers for optical thickness values 0.1, 1.0, and 10.0.

Anisotropic scattering effects again follow the same trends as the isotropic scattering cases. They are observed to become insignificant toward the outer wall.


## Acknowledgment

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